MODEL OF A VISCOUS FLUID WITH AN ANTISYMMETRIC STRESS TENSOR

(O MODELI VIAZKOI ZHIDKOSTI S NESIMMETRICHNYM TENZOROM NAPRIAZHENII)

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The Navier-Stokes equations of motion of a viscous fluid may be derived from considerations of the kinetic theory of fluids and gases for systems of low density, or consisting of spherical molecules only [1 and 2]. The stress tensor of a viscous fluid is in the general case antisymmetric. A model of a continuous medium, having an antisymmetric stress tensor, was suggested in [1]. The mechanical behavior of the medium was defined simultaneously by means of the usual velocity field, and of the field of internal spin of particles constituting a "point" of the physical continuum. An axial tensor of second rank defining the internal couple-stresses was introduced into the considerations in [1], together with the antisymmetric tensor of force-stresses. Characteristic rheology equations and equations of motion of a viscous fluid with relaxation properties were obtained in [1], on the assumption that couple-stresses exert work on the inner spin translations only. It was also shown in that work that, if the couple-stresses in a liquid or gas could be neglected, the physical properties of the medium would be defined by the usual viscosity coefficient, the coefficient of rotational viscosity, and by the relaxation time. A more general model of a structured continuum was suggested in [3 and 4].

A medium having polarizing properties was considered in [3], in which external mass-couples of an electromagnetic nature were considered together with the antisymmetric stress tensor and the couple-stresses. A detailed analysis of a model of a dielectric liquid in an electric field is given in [4].

A linear model of a polarizing dielectric is constructed in the present paper on the assumption that the stress tensor symmetric part is dependent on the symmetric part of the velocity deformation tensor only, while the tensor of couple-stresses is assumed to be symmetric, and dependent on the symmetric part of the tensor-gradient of angular velocity of the inner spin of particles only.

Real flows were considered in [4] under the condition that at their rigid boundaries either the stress tensor antisymmetric part, or the vector of inner spin angular velocity are zero.

A variant of a structured continuum is considered below. Its characteristics are, in certain respects, more general than those of the continua considered in [1 to 4].

This model is constructed on the assumption that the couple-stresses do not exert work on internal spin translations, while exerting such on external translations within the volume. In the general case, the force and couple stresses dyadics of this model are antisymmetric, the model is subject to relaxation, and is characterized by thermomechanical effects.

The model analyzed here coincides in one of its limiting cases with that of Grad's [1 and 2], while in another, it is characterized by the usual Newtonian viscosity in shear, and the viscosity at local bending - twisting.

We shall consider an isotropic material continuum at each point of which are known the translation velocity vector $\mathbf{v}$, and the vector of angular velocity of spin $\boldsymbol{\omega}$. We shall assume that a mass-force vector $\mathbf{f}$, and a vector of external mass-couple $\mathbf{c}$ are applied at each point of an arbitrary volume V. Force-stresses $\mathbf{t}_n$ and couple-stresses $\mathbf{m}_n$ act at the surface S of the volume V.

We write the equations of mass conservation, and the equations of change of momentum, of moment of momentum, and of energy, as follows:

$$\frac{d}{dt}\int_V \rho\, dV = 0, \qquad \frac{d}{dt}\int_V \rho \mathbf{v}\, dV = \int_S \mathbf{t}_n\, dS + \int_V \rho \mathbf{f}\, dV$$

$$\frac{d}{dt}\int_V (\mathbf{r}\times\mathbf{v} + J\boldsymbol{\omega})\rho\, dV = \int_S (\mathbf{r}\times\mathbf{t}_n + \mathbf{m}_n)\, dS + \int_V (\mathbf{r}\times\mathbf{f} + \mathbf{c})\rho dV$$

$$\frac{d}{dt}\int_V \left(\frac{v^2}{2} + u + \frac{J\omega^2}{2}\right)\rho\, dV = \int_S \left(\mathbf{t}_n\cdot\mathbf{v} + \frac{1}{2}\mathbf{m}_n\cdot\nabla\times\mathbf{v}\right) dS -$$

$$-\int_V \nabla\cdot\mathbf{q} dV + \int_V \left(\mathbf{f}\cdot\mathbf{v} + \frac{1}{2}\mathbf{c}\cdot\nabla\times\mathbf{v}\right)\rho\, dV \tag{1}$$

Here ρ is the density, $d(\ldots)/dt$ is the total derivative with respect to time, $\mathbf{r}$ is the line vector of a point, ∇ is the spatial gradient, $\mathbf{q}$ the heat flux vector, u the specific internal energy which is a function of state, and J is the average value of the moment of inertia at a point of the structured continuum [1 and 2]. The equation of energy change (1) has been written on the assumption that the work of the couple-stress vector $\mathbf{m}_n$, and that of vector $\mathbf{c}$ exerted on the translation of rotation $\boldsymbol{\omega}$ can be neglected. [1].

The dyadics of force-stresses $\boldsymbol{\tau}$ and couple-stresses $\boldsymbol{\mu}$ are related to the outward normal vector $\mathbf{n}$ by relations

$$\mathbf{t}_n = \mathbf{n}\cdot\boldsymbol{\tau}, \qquad \mathbf{m}_n = \mathbf{n}\cdot\boldsymbol{\mu} \tag{2}$$

Taking into account (2), we obtain from (1)

$$\begin{gathered}\rho\frac{d\mathbf{v}}{dt} = \nabla\cdot\boldsymbol{\tau} + \rho\mathbf{f}, \quad \frac{d\rho}{dt} = 0, \quad \nabla\cdot\boldsymbol{\mu} + \rho\mathbf{c} + \boldsymbol{\tau}\times\cdot\mathbf{I} = \rho J\frac{d\boldsymbol{\omega}}{dt} \\ \rho\frac{du}{dt} = \boldsymbol{\tau}\cdot\cdot\nabla\mathbf{v} - \frac{1}{2}(\boldsymbol{\tau}\times\cdot\mathbf{I})\cdot\nabla\times\mathbf{v} + \frac{1}{2}\boldsymbol{\mu}\cdot\cdot\nabla\nabla\times\mathbf{v} + \\ + \frac{1}{2}\rho J\frac{d\boldsymbol{\omega}}{dt}\cdot(\nabla\times\mathbf{q} - 2\boldsymbol{\omega}) - \nabla\cdot\mathbf{q}\end{gathered} \tag{3}$$

Here $\mathbf{I}$ is the unit dyadic, and symbol $(\times\cdot)$ indicates an operation in which the first factors of a dyadic are subject to scalar multiplication, while the left-hand ones are subject to vector multiplication [5].

We shall represent dyadics $\boldsymbol{\tau}$, $\nabla\mathbf{v}$, $\nabla\nabla\times\mathbf{v}$, $\boldsymbol{\mu}$ in the following form:

$$\tau = (\pi^* - p)\,\mathbf{I} + \pi^a + \pi^d, \qquad \mu = \mu^*\mathbf{I} + \mu^a + \mu^d, \qquad \tau = -p\mathbf{I} + \pi$$
$$\nabla\nabla \times \mathbf{v} = (\nabla\nabla \times \mathbf{v})^a + (\nabla\nabla \times \mathbf{v})^d, \qquad \nabla\mathbf{v} = v^*\mathbf{I} + (\nabla\mathbf{v})^a + (\nabla\mathbf{v})^d \tag{4}$$
$$\mu^* = {}^1/_3\mu\cdot\cdot\mathbf{I}, \qquad \pi^* = {}^1/_3\pi\cdot\cdot\mathbf{I}, \qquad v^* = {}^1/_3(\nabla\cdot\mathbf{v})\,\mathbf{I}$$

Here, p is the equilibrium pressure, and superscripts d and a denote symmetric and antisymmetric dyadics respectively.

Using (4) and the relationship $(\tau \times .\mathbf{I})\cdot\nabla\times\mathbf{v} = 2\pi^a\cdot\cdot\nabla\mathbf{v}$, we transform Equation (1) of energy change, and obtain

$$\rho\frac{du}{dt} = (\pi^* - p)\,\nabla\cdot\mathbf{v} + \pi^d\cdot\cdot(\nabla\mathbf{v})^d + \frac{1}{2}\mu^d\cdot\cdot(\nabla\nabla\times\mathbf{v})^d +$$
$$+\frac{1}{2}\mu^a\cdot\cdot(\nabla\nabla\times\mathbf{v})^a + \frac{1}{2}\rho J\frac{d\omega}{dt}\cdot(\nabla\times\mathbf{v} - 2\omega) - \nabla\cdot\mathbf{q} \tag{5}$$

With the use of the Gibbs thermodynamic relation, and of the mass conservation law, as given in [2], in the form

$$T\frac{ds}{dt} = \frac{du}{dt} + p\frac{dv}{dt}, \qquad \rho\frac{dv}{dt} = \nabla\cdot\mathbf{v} \tag{6}$$

where T is the absolute temperature, s the specific entropy, and v the specific volume ($v = \rho^{-1}$).

The entropy balance, obtained from (5) and (6), is expressed by

$$\rho\frac{ds}{dt} = -\nabla\cdot\frac{\mathbf{q}}{T} + \sigma \tag{7}$$

where the positive value of evolution of entropy σ is expressed by

$$\sigma = -\mathbf{q}\cdot\frac{\nabla T}{T^2} + \frac{\pi^*(\nabla\cdot\mathbf{v})}{T} + \frac{\pi^d\cdot\cdot(\nabla\mathbf{v})^d}{T} + \frac{\mu^d\cdot\cdot(\nabla\nabla\times\mathbf{v})^d}{2T} +$$
$$+\frac{\mu^a\cdot\cdot(\nabla\nabla\times\mathbf{v})^a}{2T} + \frac{1}{2T}\left[\rho J\frac{d\omega}{dt}\cdot(\nabla\times\mathbf{v} - 2\omega)\right] \geqslant 0 \tag{8}$$

The thermodynamic forces [6] in Equation (8) are as follows : the true scalar $\nabla\mathbf{v}$, the symmetric dyadic $(\nabla\mathbf{v})^d$, the symmetric pseudodyadic $(\nabla\nabla\times\mathbf{v})^d$, the antisymmetric pseudodyadic $(\nabla\nabla\times\mathbf{v})^a$, the pseudovector $(\nabla\times\mathbf{v} - 2\omega)$ and the true vector ∇T. We note that pseudovector $(\nabla\times\mathbf{v} - 2\omega)$ and pseudodyadic $(\nabla\nabla\times\mathbf{v})^d$ are axial, while the vector equivalent of the antisymmetric pseudodyadic $(\nabla\nabla\times\mathbf{v})^a$, the symmetric dyadic $(\nabla\mathbf{v})^{d'}$, and vector ∇T are polar. The linear interdependence between the thermodynamic forces and fluxes is found from (8) by using the Curie principle and the Onzager reciprocity relationships [2], in the form of

$$\mathbf{q} = -\varkappa\nabla T + T\varkappa_{12}\mathbf{b}, \qquad \pi^d = 2\eta\,(\nabla\mathbf{v})^d, \qquad \pi^* = \eta^*\nabla\cdot\mathbf{v}$$
$$\mathbf{d} = \varkappa_{12}\nabla T + c_a\mathbf{b}, \quad \mu^d = c_d(\nabla\nabla\mathbf{v})^d, \quad \rho J\frac{d\omega}{dt} = 2\eta_r(\nabla\times\mathbf{v} - 2\omega) \tag{9}$$

Here, $\mathbf{b}$ and $\mathbf{d}$ are the equivalent vectors of dyadics $(\nabla\nabla\mathbf{v})^a$, and μ^a respectively. It will be seen from (9) that the continuous medium model considered here is characterized by thermomechanichal effects resulting from the asymmetry of the couple-stress dyadic μ.

In those cases in which the thermomechanical and compressibility effects may be neglected, Equation (9) is written as follows :

$$\mathbf{q} = -\varkappa\nabla T, \quad \boldsymbol{\pi}^d = 2\eta(\nabla\mathbf{v})^d, \quad \boldsymbol{\mu}^a = c_a(\nabla\nabla\times\mathbf{v})^a$$
$$\boldsymbol{\mu}^d = c_d(\nabla\nabla\times\mathbf{v})^d, \qquad \rho J\frac{d\boldsymbol{\omega}}{dt} = 2\eta_r(\nabla\times\mathbf{v} - 2\boldsymbol{\omega}) \tag{10}$$

Here, the values of scalars η, η_r, c_a, c_d, $\varkappa$ and J are all positive. Noting that $2\boldsymbol{\tau}^a = -I\times(\boldsymbol{\tau}\times\cdot\mathbf{I})$, we obtain from (3)

$$\boldsymbol{\tau}^a = \frac{1}{2}\mathbf{I}\times\left(\nabla\cdot\boldsymbol{\mu} + \rho\mathbf{c} - \rho J\frac{d\boldsymbol{\omega}}{dt}\right) \tag{11}$$

Using (9) to (11) and (4), we find from (5) that

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p + 2\eta\nabla\cdot(\nabla\mathbf{v})^d + \frac{c_d}{2}\nabla\times\nabla\cdot(\nabla\nabla\times\mathbf{v})^d +$$
$$+\frac{c_a}{2}\nabla\times\nabla\cdot(\nabla\nabla\times\mathbf{v})^a + \eta_r\nabla\times(2\boldsymbol{\omega} - \nabla\times\mathbf{v}) + \rho\mathbf{f} \tag{12}$$
$$\frac{d\boldsymbol{\omega}}{dt} = \frac{2\eta_r}{\rho J}(\nabla\times\mathbf{v} - 2\boldsymbol{\omega}), \qquad \nabla\cdot\mathbf{v} = 0$$

The thermal conductivity equation of an incompressible medium is

$$\rho c_p\left[\frac{\partial T}{\partial t} + \mathbf{v}\cdot(\nabla T)\right] = -T\nabla\cdot\frac{\mathbf{q}}{T} + T\sigma \tag{13}$$

where C_p is the specific heat at constant pressure, and σ is defined by (8) and (9).

For $C_d = C_a = 0$, and $c = 0$, Equations (12) coincide with the equations of motion of an incompressible viscous medium [2], for which the stress dyadic is antisymmetric, due to the inner spin of particles

$$\boldsymbol{\tau}^a = -\frac{\rho J}{2}\mathbf{I}\times\frac{d\boldsymbol{\omega}}{dt}$$

With $J = 0$, Equations (10), (12) and (13) together with relationship

$$\boldsymbol{\tau}^a = \frac{1}{2}\mathbf{I}\times(\nabla\cdot\boldsymbol{\mu}^d + \nabla\cdot\boldsymbol{\mu}^a + \mu^* + \rho\mathbf{c}) \tag{14}$$

define an incompressible viscous fluid flow in which the stress dyadic asymmetry is the result of presence of couple-stresses.

When $J = 0$, then Equations (10), (12) and (14) become analogous to those of the linearized theory of an elastic medium in the presence of couple-stresses [5].

The model of the viscous fluid considered here is, in the general case, characterized by the usual Newtonian resistance to shear, the resistance to local bending - twisting [5], by its relaxation properties [1 and 2], and by thermomechanical phenomena.

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